Encoding Collatz-like problems into termination of string rewriting

Emre Yolcu

Computer Science Department
Carnegie Mellon University

with Scott Aaronson and Marijn Heule

eyolcu@cs.cmu.edu

Collatz conjecture

$$C(n) = \begin{cases} n/2 & \text{if } n \equiv 0 \pmod{2} \\ 3n+1 & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

Conjecture

For all $n \in \mathbb{N}^+$*, the trajectory* $n, C(n), C(C(n)), \ldots$ *reaches* 1*.*

[1] https://pcbarina.fit.vutbr.cz/

Collatz conjecture

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Conjecture

For all $n \in \mathbb{N}^+$, the trajectory $n, C(n), C(C(n)), \ldots$ reaches 1.

Verified for all n below 955×2^{60} as of today.[1]

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String rewriting

Σ	alphabet
$\ell \to r$	rewrite rule
R	rewriting system
$\to_R$	rewrite relation

Example: $R = \{ab \to ba\}$

$$\underline{ab}bab \to_R b\underline{ab}ab \to_R bba\underline{ab} \to_R bb\underline{ab}a \to_R bbbaa$$

Termination

Definition

R is *terminating* if there is no infinite chain

$$X_0 \quad \to_R \quad X_1 \quad \to_R \quad X_2 \quad \to_R \quad \cdots$$

where $X_i \in \Sigma^*$.

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where $X_i \in \Sigma^*$.

Theorem

R is terminating iff there is a well-founded order $\succ$ such that for all $X, Y \in \Sigma^*$, if $X \to_R Y$ then $X \succ Y$.

Interpretation method [Manna and Ness, 1970]

To prove that a rewriting system R with alphabet Σ is terminating:

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1. Fix some domain A equipped with a well-founded order $\succ$.
2. Translate each $s \in \Sigma$ into a monotone function $[s] \colon A \to A$ such that for each $\ell \to r \in R$,

$$[\ell](x) \succ [r](x) \text{ for all } x \in A,$$

   where for a string $T = t_1 \ldots t_n$ we define $[T] := [t_1] \circ \cdots \circ [t_n]$.

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Example: $R = \{ab \to ba\}$. Fix the domain $(A, \succ)$ to be $(\mathbb{N}^+, >)$.

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Example: $R = \{ab \to ba\}$. Fix the domain $(A, \succ)$ to be $(\mathbb{N}^+, >)$. With the interpretations $[a](x) = x^2$ and $[b](x) = x + 1$,

$$[ab](x) = (x+1)^2 > x^2 + 1 = [ba](x) \text{ for all } x \in \mathbb{N}^+.$$

Matrix interpretations [Hofbauer and Waldmann, 2006]

To prove that a rewriting system R with alphabet Σ is terminating:

1. Fix A to be $\mathbb{N}^d$ and equip it with the well-founded order $\succ$ defined as

$$\mathbf{x} \succ \mathbf{y} \quad \Longleftrightarrow \quad x_1 > y_1 \quad \wedge \quad x_i \geq y_i \text{ for } i \in \{2, 3, \ldots, d\}.$$

Matrix interpretations [Hofbauer and Waldmann, 2006]

To prove that a rewriting system R with alphabet Σ is terminating:

1. Fix A to be $\mathbb{N}^d$ and equip it with the well-founded order $\succ$ defined as

$$\mathbf{x} \succ \mathbf{y} \quad \iff \quad x_1 > y_1 \quad \wedge \quad x_i \geq y_i \text{ for } i \in \{2, 3, \ldots, d\}.$$

2. Look for an affine function $[s]\colon \mathbb{N}^d \to \mathbb{N}^d$ for each $s \in \Sigma$ (i.e., matrices $\mathbf{M}_s$ and vectors $\mathbf{v}_s$) such that for each $\ell \to r \in R$,

$$[\ell](\mathbf{x}) = \mathbf{M}_\ell \mathbf{x} + \mathbf{v}_\ell \succ \mathbf{M}_r \mathbf{x} + \mathbf{v}_r = [r](\mathbf{x}) \text{ for all } \mathbf{x} \in \mathbb{N}^d.$$

Example proof

$$Z = \left\{ \begin{matrix} aa \to bc \\ bb \to ac \\ cc \to ab \end{matrix} \right\}$$

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$$Z = \left\{ \begin{array}{l} aa \to bc \\ bb \to ac \\ cc \to ab \end{array} \right\}$$

$$[a](\mathbf{x}) = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[b](\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$[c](\mathbf{x}) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Example proof (cont.)

$$\mathbf{x} \succ \mathbf{y} \quad \Longleftrightarrow \quad x_1 > y_1 \quad \wedge \quad x_i \geq y_i \text{ for } i \in \{2, 3, \ldots, d\}.$$

It is decidable to check that for all $\mathbf{x}$,

$$[aa](\mathbf{x}) = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad \succ \quad \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = [bc](\mathbf{x}),$$

$$[bb](\mathbf{x}) = \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4 \\ 0 \\ 2 \\ 6 \end{bmatrix} \quad \succ \quad \begin{bmatrix} 1 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 6 \end{bmatrix} = [ac](\mathbf{x}),$$

$$[cc](\mathbf{x}) = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} \quad \succ \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} = [ab](\mathbf{x}).$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{rcl} \mathtt{h11} & \to & \mathtt{1h} \end{array} \qquad \begin{array}{rcl} \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\ \mathtt{1s} & \to & \mathtt{s1} \\ \diamond\mathtt{s} & \to & \diamond\mathtt{h} \end{array} \qquad \begin{array}{rcl} \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\ \mathtt{1t} & \to & \mathtt{t111} \\ \diamond\mathtt{t} & \to & \diamond\mathtt{h} \end{array}$$

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System $\mathcal{U}$:

$$\mathtt{h11} \to \mathtt{1h} \qquad \mathtt{11h}\diamond \to \mathtt{11s}\diamond \qquad \mathtt{h1}\diamond \to \mathtt{t11}\diamond$$

$$\mathtt{1s} \to \mathtt{s1} \qquad \mathtt{1t} \to \mathtt{t111}$$

$$\diamond\mathtt{s} \to \diamond\mathtt{h} \qquad \diamond\mathtt{t} \to \diamond\mathtt{h}$$

Rewrite sequences:

$$\diamond\mathtt{h}\mathtt{1}^{2n}\diamond \;\to\; \cdots \;\to\; \diamond\mathtt{h}\mathtt{1}^{n}\diamond \qquad \text{for } n > 1$$

$$\diamond\mathtt{h}\mathtt{1}^{2n+1}\diamond \;\to\; \cdots \;\to\; \diamond\mathtt{h}\mathtt{1}^{3n+2}\diamond \qquad \text{for } n \geq 0$$

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Example run:

$$\diamond\mathtt{h111}\diamond \qquad (3)$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{rcl} \mathtt{h11} & \to & \mathtt{1h} \end{array} \qquad \begin{array}{rcl} \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\ \mathtt{1s} & \to & \mathtt{s1} \\ \diamond\mathtt{s} & \to & \diamond\mathtt{h} \end{array} \qquad \begin{array}{rcl} \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\ \mathtt{1t} & \to & \mathtt{t111} \\ \diamond\mathtt{t} & \to & \diamond\mathtt{h} \end{array}$$

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Example run:

$$\diamond\mathtt{1h1}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\mathtt{h11} \to \mathtt{1h} \qquad \begin{aligned} \mathtt{11h}\diamond &\to \mathtt{11s}\diamond \\ \mathtt{1s} &\to \mathtt{s1} \\ \diamond\mathtt{s} &\to \diamond\mathtt{h} \end{aligned} \qquad \begin{aligned} \mathtt{h1}\diamond &\to \mathtt{t11}\diamond \\ \mathtt{1t} &\to \mathtt{t111} \\ \diamond\mathtt{t} &\to \diamond\mathtt{h} \end{aligned}$$

Rewrite sequences:

$$\diamond\mathtt{h}\mathtt{1}^{2n}\diamond \;\to\; \cdots \;\to\; \diamond\mathtt{h}\mathtt{1}^{n}\diamond \qquad \text{for } n > 1$$

$$\diamond\mathtt{h}\mathtt{1}^{2n+1}\diamond \;\to\; \cdots \;\to\; \diamond\mathtt{h}\mathtt{1}^{3n+2}\diamond \qquad \text{for } n \geq 0$$

Example run:

$$\diamond\mathtt{1t11}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$
\begin{array}{rcl}
\mathtt{h11} & \to & \mathtt{1h}
\end{array}
\qquad
\begin{array}{rcl}
\mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\
\mathtt{1s} & \to & \mathtt{s1} \\
\diamond\mathtt{s} & \to & \diamond\mathtt{h}
\end{array}
\qquad
\begin{array}{rcl}
\mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\
\mathtt{1t} & \to & \mathtt{t111} \\
\diamond\mathtt{t} & \to & \diamond\mathtt{h}
\end{array}
$$

Rewrite sequences:

$$
\begin{array}{rcccll}
\diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\
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\end{array}
$$

Example run:

$$\diamond\mathtt{t11111}\diamond$$

Unary encoding [Zantema, 2005]

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$$
\begin{array}{lll}
\mathtt{h11} \to \mathtt{1h} & \mathtt{11h}\diamond \to \mathtt{11s}\diamond & \mathtt{h1}\diamond \to \mathtt{t11}\diamond \\
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 & \diamond\mathtt{s} \to \diamond\mathtt{h} & \diamond\mathtt{t} \to \diamond\mathtt{h}
\end{array}
$$

Rewrite sequences:

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\end{array}
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Example run:

$$\diamond\mathtt{h11111}\diamond \qquad (\mathbf{5})$$

Unary encoding [Zantema, 2005]

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$$\begin{array}{rcl} \mathtt{h11} & \to & \mathtt{1h} \end{array} \qquad \begin{array}{rcl} \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\ \mathtt{1s} & \to & \mathtt{s1} \\ \diamond\mathtt{s} & \to & \diamond\mathtt{h} \end{array} \qquad \begin{array}{rcl} \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\ \mathtt{1t} & \to & \mathtt{t111} \\ \diamond\mathtt{t} & \to & \diamond\mathtt{h} \end{array}$$

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$$\begin{array}{rcccccl} \diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & & \text{for } n \geq 0 \end{array}$$

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Unary encoding [Zantema, 2005]

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Rewrite sequences:

$$\begin{array}{rcccccl} \diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & & \text{for } n \geq 0 \end{array}$$

Example run:

$$\diamond\mathtt{11h1}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$
\begin{array}{lll}
\mathtt{h11} \to \mathtt{1h} & \mathtt{11h}\diamond \to \mathtt{11s}\diamond & \mathtt{h1}\diamond \to \mathtt{t11}\diamond \\
 & \mathtt{1s} \to \mathtt{s1} & \mathtt{1t} \to \mathtt{t111} \\
 & \diamond\mathtt{s} \to \diamond\mathtt{h} & \diamond\mathtt{t} \to \diamond\mathtt{h}
\end{array}
$$

Rewrite sequences:

$$
\begin{array}{lllll}
\diamond\mathtt{h1}^{2n}\diamond & \to \cdots \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\
\diamond\mathtt{h1}^{2n+1}\diamond & \to \cdots \to & \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0
\end{array}
$$

Example run:

$$\diamond\mathtt{11t11}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$
\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\mathtt{h11} & \to & \mathtt{1h} & \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond & \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\
 & & & \mathtt{1s} & \to & \mathtt{s1} & \mathtt{1t} & \to & \mathtt{t111} \\
 & & & \diamond\mathtt{s} & \to & \diamond\mathtt{h} & \diamond\mathtt{t} & \to & \diamond\mathtt{h}
\end{array}
$$

Rewrite sequences:

$$
\begin{array}{rcccll}
\diamond\mathtt{h}\mathtt{1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h}\mathtt{1}^{n}\diamond & \text{for } n > 1 \\
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\end{array}
$$

Example run:

$$\diamond\mathtt{1t11111}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$
\begin{array}{rcl}
\mathtt{h11} & \to & \mathtt{1h}
\end{array}
\qquad
\begin{array}{rcl}
\mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\
\mathtt{1s} & \to & \mathtt{s1} \\
\diamond\mathtt{s} & \to & \diamond\mathtt{h}
\end{array}
\qquad
\begin{array}{rcl}
\mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\
\mathtt{1t} & \to & \mathtt{t111} \\
\diamond\mathtt{t} & \to & \diamond\mathtt{h}
\end{array}
$$

Rewrite sequences:

$$
\begin{array}{rcccll}
\diamond\mathtt{h}\mathtt{1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h}\mathtt{1}^{n}\diamond & \text{for } n > 1 \\
\diamond\mathtt{h}\mathtt{1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h}\mathtt{1}^{3n+2}\diamond & \text{for } n \geq 0
\end{array}
$$

Example run:

$$\diamond\mathtt{t1111111}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{rcl} \mathtt{h11} & \to & \mathtt{1h} \end{array} \qquad \begin{array}{rcl} \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\ \mathtt{1s} & \to & \mathtt{s1} \\ \diamond\mathtt{s} & \to & \diamond\mathtt{h} \end{array} \qquad \begin{array}{rcl} \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\ \mathtt{1t} & \to & \mathtt{t111} \\ \diamond\mathtt{t} & \to & \diamond\mathtt{h} \end{array}$$

Rewrite sequences:

$$\begin{array}{rcccccl} \diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & & \text{for } n \geq 0 \end{array}$$

Example run:

$$\diamond\mathtt{h11111111}\diamond \qquad (8)$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$
\begin{array}{lll}
\mathtt{h11} \to \mathtt{1h} & \mathtt{11h}\diamond \to \mathtt{11s}\diamond & \mathtt{h1}\diamond \to \mathtt{t11}\diamond \\
& \mathtt{1s} \to \mathtt{s1} & \mathtt{1t} \to \mathtt{t111} \\
& \diamond\mathtt{s} \to \diamond\mathtt{h} & \diamond\mathtt{t} \to \diamond\mathtt{h}
\end{array}
$$

Rewrite sequences:

$$
\begin{array}{lllll}
\diamond\mathtt{h1}^{2n}\diamond & \to \cdots \to & \diamond\mathtt{h1}^{n}\diamond & & \text{for } n > 1 \\
\diamond\mathtt{h1}^{2n+1}\diamond & \to \cdots \to & \diamond\mathtt{h1}^{3n+2}\diamond & & \text{for } n \geq 0
\end{array}
$$

Example run:

$$\diamond\mathtt{1h111111}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{rcl} \mathtt{h11} & \to & \mathtt{1h} \end{array} \qquad \begin{array}{rcl} \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\ \mathtt{1s} & \to & \mathtt{s1} \\ \diamond\mathtt{s} & \to & \diamond\mathtt{h} \end{array} \qquad \begin{array}{rcl} \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\ \mathtt{1t} & \to & \mathtt{t111} \\ \diamond\mathtt{t} & \to & \diamond\mathtt{h} \end{array}$$

Rewrite sequences:

$$\begin{array}{rccccl} \diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0 \end{array}$$

Example run:

$$\diamond\mathtt{11h1111}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{rcl} \mathtt{h11} & \to & \mathtt{1h} \end{array} \qquad \begin{array}{rcl} \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\ \mathtt{1s} & \to & \mathtt{s1} \\ \diamond\mathtt{s} & \to & \diamond\mathtt{h} \end{array} \qquad \begin{array}{rcl} \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\ \mathtt{1t} & \to & \mathtt{t111} \\ \diamond\mathtt{t} & \to & \diamond\mathtt{h} \end{array}$$

Rewrite sequences:

$$\begin{array}{rcccccl} \diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & & \text{for } n \geq 0 \end{array}$$

Example run:

$$\diamond\mathtt{111h11}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$
\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\mathtt{h11} & \to & \mathtt{1h} & \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond & \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\
 & & & \mathtt{1s} & \to & \mathtt{s1} & \mathtt{1t} & \to & \mathtt{t111} \\
 & & & \diamond\mathtt{s} & \to & \diamond\mathtt{h} & \diamond\mathtt{t} & \to & \diamond\mathtt{h}
\end{array}
$$

Rewrite sequences:

$$
\begin{array}{rcccl@{\qquad}l}
\diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\
\diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0
\end{array}
$$

Example run:

$$\diamond\mathtt{1111h}\diamond$$

Unary encoding [Zantema, 2005]

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\diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\
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\end{array}
$$

Example run:

$$\diamond\mathtt{1111s}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$
\begin{array}{rcl}
\mathtt{h11} & \to & \mathtt{1h}
\end{array}
\qquad
\begin{array}{rcl}
\mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\
\mathtt{1s} & \to & \mathtt{s1} \\
\diamond\mathtt{s} & \to & \diamond\mathtt{h}
\end{array}
\qquad
\begin{array}{rcl}
\mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\
\mathtt{1t} & \to & \mathtt{t111} \\
\diamond\mathtt{t} & \to & \diamond\mathtt{h}
\end{array}
$$

Rewrite sequences:

$$
\begin{array}{rcccll}
\diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\
\diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0
\end{array}
$$

Example run:

$$\diamond\mathtt{111s1}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{rcl} \mathtt{h11} & \to & \mathtt{1h} \end{array} \qquad \begin{array}{rcl} \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\ \mathtt{1s} & \to & \mathtt{s1} \\ \diamond\mathtt{s} & \to & \diamond\mathtt{h} \end{array} \qquad \begin{array}{rcl} \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\ \mathtt{1t} & \to & \mathtt{t111} \\ \diamond\mathtt{t} & \to & \diamond\mathtt{h} \end{array}$$

Rewrite sequences:

$$\begin{array}{rcccccl} \diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & & \text{for } n \geq 0 \end{array}$$

Example run:

$$\diamond\mathtt{11s11}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{lll} \mathtt{h11} \to \mathtt{1h} & \mathtt{11h}\diamond \to \mathtt{11s}\diamond & \mathtt{h1}\diamond \to \mathtt{t11}\diamond \\ & \mathtt{1s} \to \mathtt{s1} & \mathtt{1t} \to \mathtt{t111} \\ & \diamond\mathtt{s} \to \diamond\mathtt{h} & \diamond\mathtt{t} \to \diamond\mathtt{h} \end{array}$$

Rewrite sequences:

$$\begin{array}{lllllr} \diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0 \end{array}$$

Example run:

$$\diamond\mathtt{1s111}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{rcl@{\qquad}rcl@{\qquad}rcl}
\mathtt{h11} & \to & \mathtt{1h} & \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond & \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\
 & & & \mathtt{1s} & \to & \mathtt{s1} & \mathtt{1t} & \to & \mathtt{t111} \\
 & & & \diamond\mathtt{s} & \to & \diamond\mathtt{h} & \diamond\mathtt{t} & \to & \diamond\mathtt{h}
\end{array}$$

Rewrite sequences:

$$\begin{array}{rcccccl}
\diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & & \text{for } n > 1 \\
\diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & & \text{for } n \geq 0
\end{array}$$

Example run:

$$\diamond\mathtt{s1111}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$
\begin{array}{rclcrclcrcl}
\mathtt{h11} & \to & \mathtt{1h} & \quad & \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond & \quad & \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\
 & & & & \mathtt{1s} & \to & \mathtt{s1} & & \mathtt{1t} & \to & \mathtt{t111} \\
 & & & & \diamond\mathtt{s} & \to & \diamond\mathtt{h} & & \diamond\mathtt{t} & \to & \diamond\mathtt{h}
\end{array}
$$

Rewrite sequences:

$$
\begin{array}{rccccl}
\diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\
\diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0
\end{array}
$$

Example run:

$$\diamond\mathtt{h1111}\diamond \qquad (4)$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{aligned} \mathtt{h11} &\to \mathtt{1h} \end{aligned} \qquad \begin{aligned} \mathtt{11h}\diamond &\to \mathtt{11s}\diamond \\ \mathtt{1s} &\to \mathtt{s1} \\ \diamond\mathtt{s} &\to \diamond\mathtt{h} \end{aligned} \qquad \begin{aligned} \mathtt{h1}\diamond &\to \mathtt{t11}\diamond \\ \mathtt{1t} &\to \mathtt{t111} \\ \diamond\mathtt{t} &\to \diamond\mathtt{h} \end{aligned}$$

Rewrite sequences:

$$\begin{array}{rcccll} \diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0 \end{array}$$

Example run:

$$\diamond\mathtt{1h11}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{aligned} \mathtt{h11} &\to \mathtt{1h} & \mathtt{11h}\diamond &\to \mathtt{11s}\diamond & \mathtt{h1}\diamond &\to \mathtt{t11}\diamond \\ & & \mathtt{1s} &\to \mathtt{s1} & \mathtt{1t} &\to \mathtt{t111} \\ & & \diamond\mathtt{s} &\to \diamond\mathtt{h} & \diamond\mathtt{t} &\to \diamond\mathtt{h} \end{aligned}$$

Rewrite sequences:

$$\begin{aligned} \diamond\mathtt{h1}^{2n}\diamond \quad &\to \quad \cdots \quad \to \quad \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond \quad &\to \quad \cdots \quad \to \quad \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0 \end{aligned}$$

Example run:

$$\diamond\mathtt{11h}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{rcl} \mathtt{h11} & \to & \mathtt{1h} \end{array} \qquad \begin{array}{rcl} \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\ \mathtt{1s} & \to & \mathtt{s1} \\ \diamond\mathtt{s} & \to & \diamond\mathtt{h} \end{array} \qquad \begin{array}{rcl} \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\ \mathtt{1t} & \to & \mathtt{t111} \\ \diamond\mathtt{t} & \to & \diamond\mathtt{h} \end{array}$$

Rewrite sequences:

$$\begin{array}{rcccll} \diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0 \end{array}$$

Example run:

$$\diamond\mathtt{11s}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{rcl} \mathtt{h11} & \to & \mathtt{1h} \end{array} \qquad \begin{array}{rcl} \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond \\ \mathtt{1s} & \to & \mathtt{s1} \\ \diamond\mathtt{s} & \to & \diamond\mathtt{h} \end{array} \qquad \begin{array}{rcl} \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\ \mathtt{1t} & \to & \mathtt{t111} \\ \diamond\mathtt{t} & \to & \diamond\mathtt{h} \end{array}$$

Rewrite sequences:

$$\begin{array}{rcccll} \diamond\mathtt{h}\mathtt{1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h}\mathtt{1}^{n}\diamond & \text{for } n > 1 \\ \diamond\mathtt{h}\mathtt{1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h}\mathtt{1}^{3n+2}\diamond & \text{for } n \geq 0 \end{array}$$

Example run:

$$\diamond\mathtt{1s1}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\begin{array}{rclcrclcrcl} \mathtt{h11} & \to & \mathtt{1h} & \quad & \mathtt{11h}\diamond & \to & \mathtt{11s}\diamond & \quad & \mathtt{h1}\diamond & \to & \mathtt{t11}\diamond \\ & & & & \mathtt{1s} & \to & \mathtt{s1} & & \mathtt{1t} & \to & \mathtt{t111} \\ & & & & \diamond\mathtt{s} & \to & \diamond\mathtt{h} & & \diamond\mathtt{t} & \to & \diamond\mathtt{h} \end{array}$$

Rewrite sequences:

$$\begin{array}{rcccll} \diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\ \diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0 \end{array}$$

Example run:

$$\diamond\mathtt{s11}\diamond$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$\mathtt{h11} \to \mathtt{1h} \qquad \mathtt{11h}\diamond \to \mathtt{11s}\diamond \qquad \mathtt{h1}\diamond \to \mathtt{t11}\diamond$$

$$\mathtt{1s} \to \mathtt{s1} \qquad \mathtt{1t} \to \mathtt{t111}$$

$$\diamond\mathtt{s} \to \diamond\mathtt{h} \qquad \diamond\mathtt{t} \to \diamond\mathtt{h}$$

Rewrite sequences:

$$\diamond\mathtt{h1}^{2n}\diamond \to \cdots \to \diamond\mathtt{h1}^{n}\diamond \qquad \text{for } n > 1$$

$$\diamond\mathtt{h1}^{2n+1}\diamond \to \cdots \to \diamond\mathtt{h1}^{3n+2}\diamond \qquad \text{for } n \geq 0$$

Example run:

$$\diamond\mathtt{h11}\diamond \qquad (2)$$

Unary encoding [Zantema, 2005]

System $\mathcal{U}$:

$$
\begin{array}{lll}
\mathtt{h11} \to \mathtt{1h} & \mathtt{11h}\diamond \to \mathtt{11s}\diamond & \mathtt{h1}\diamond \to \mathtt{t11}\diamond \\
 & \mathtt{1s} \to \mathtt{s1} & \mathtt{1t} \to \mathtt{t111} \\
 & \diamond\mathtt{s} \to \diamond\mathtt{h} & \diamond\mathtt{t} \to \diamond\mathtt{h}
\end{array}
$$

Rewrite sequences:

$$
\begin{array}{llllll}
\diamond\mathtt{h1}^{2n}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{n}\diamond & \text{for } n > 1 \\
\diamond\mathtt{h1}^{2n+1}\diamond & \to & \cdots & \to & \diamond\mathtt{h1}^{3n+2}\diamond & \text{for } n \geq 0
\end{array}
$$

Example run:

$$\diamond\mathtt{1h}\diamond \qquad (1)$$

Mixed-base encoding

System $\mathcal{M}$:

$$
\begin{array}{llll}
\mathtt{f}\triangleright \to \triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \to \mathtt{2}\triangleright & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}
$$

Mixed-base encoding

System $\mathcal{M}$:

$$
\begin{array}{llll}
\mathtt{f}\triangleright \rightarrow \triangleright & \mathtt{f0} \rightarrow \mathtt{0f} & \mathtt{t0} \rightarrow \mathtt{1t} & \triangleleft\mathtt{0} \rightarrow \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \rightarrow \mathtt{2}\triangleright & \mathtt{f1} \rightarrow \mathtt{0t} & \mathtt{t1} \rightarrow \mathtt{2f} & \triangleleft\mathtt{1} \rightarrow \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \rightarrow \mathtt{1f} & \mathtt{t2} \rightarrow \mathtt{2t} & \triangleleft\mathtt{2} \rightarrow \triangleleft\mathtt{ft}
\end{array}
$$

Digits:

$$
\begin{array}{lcl}
\mathtt{f} & = & 0_2 \\
\mathtt{t} & = & 1_2
\end{array}
\qquad
\begin{array}{lcl}
\mathtt{0} & = & 0_3 \\
\mathtt{1} & = & 1_3 \\
\mathtt{2} & = & 2_3
\end{array}
\qquad
\begin{array}{lcl}
\triangleleft & = & 1_0 \\
\triangleright & = & 0_1
\end{array}
$$

Mixed-base encoding

System $\mathcal{M}$:

$$\begin{array}{llll}
\texttt{f}\triangleright \to \triangleright & \texttt{f0} \to \texttt{0f} & \texttt{t0} \to \texttt{1t} & \triangleleft\texttt{0} \to \triangleleft\texttt{t} \\
\texttt{t}\triangleright \to \texttt{2}\triangleright & \texttt{f1} \to \texttt{0t} & \texttt{t1} \to \texttt{2f} & \triangleleft\texttt{1} \to \triangleleft\texttt{ff} \\
 & \texttt{f2} \to \texttt{1f} & \texttt{t2} \to \texttt{2t} & \triangleleft\texttt{2} \to \triangleleft\texttt{ft}
\end{array}$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$\begin{aligned} \texttt{f}(x) &= 2x \\ \texttt{t}(x) &= 2x+1 \end{aligned} \qquad \begin{aligned} \texttt{0}(x) &= 3x \\ \texttt{1}(x) &= 3x+1 \\ \texttt{2}(x) &= 3x+2 \end{aligned} \qquad \begin{aligned} \triangleleft(x) &= 1 \\ \triangleright(x) &= x \end{aligned}$$

Mixed-base encoding

System $\mathcal{M}$:

$$\begin{array}{llll}
\texttt{f}\triangleright \to \triangleright & \texttt{f0} \to \texttt{0f} & \texttt{t0} \to \texttt{1t} & \triangleleft\texttt{0} \to \triangleleft\texttt{t} \\
\texttt{t}\triangleright \to \texttt{2}\triangleright & \texttt{f1} \to \texttt{0t} & \texttt{t1} \to \texttt{2f} & \triangleleft\texttt{1} \to \triangleleft\texttt{ff} \\
 & \texttt{f2} \to \texttt{1f} & \texttt{t2} \to \texttt{2t} & \triangleleft\texttt{2} \to \triangleleft\texttt{ft}
\end{array}$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$\begin{aligned} \texttt{f}(x) &= 2x \\ \texttt{t}(x) &= 2x+1 \end{aligned} \qquad \begin{aligned} \texttt{0}(x) &= 3x \\ \texttt{1}(x) &= 3x+1 \\ \texttt{2}(x) &= 3x+2 \end{aligned} \qquad \begin{aligned} \triangleleft(x) &= 1 \\ \triangleright(x) &= x \end{aligned}$$

Example run:

$$\triangleleft 0 \triangleright \qquad\qquad (1_0 0_3 0_1 = \mathbf{3})$$

Mixed-base encoding

System $\mathcal{M}$:

$$\begin{array}{lll} \mathtt{f}\triangleright & \to & \triangleright \\ \mathtt{t}\triangleright & \to & \mathtt{2}\triangleright \end{array} \qquad \begin{array}{lll} \mathtt{f0} & \to & \mathtt{0f} \\ \mathtt{f1} & \to & \mathtt{0t} \\ \mathtt{f2} & \to & \mathtt{1f} \end{array} \qquad \begin{array}{lll} \mathtt{t0} & \to & \mathtt{1t} \\ \mathtt{t1} & \to & \mathtt{2f} \\ \mathtt{t2} & \to & \mathtt{2t} \end{array} \qquad \begin{array}{lll} \triangleleft\mathtt{0} & \to & \triangleleft\mathtt{t} \\ \triangleleft\mathtt{1} & \to & \triangleleft\mathtt{ff} \\ \triangleleft\mathtt{2} & \to & \triangleleft\mathtt{ft} \end{array}$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$\begin{array}{rcl} \mathtt{f}(x) & = & 2x \\ \mathtt{t}(x) & = & 2x+1 \end{array} \qquad \begin{array}{rcl} \mathtt{0}(x) & = & 3x \\ \mathtt{1}(x) & = & 3x+1 \\ \mathtt{2}(x) & = & 3x+2 \end{array} \qquad \begin{array}{rcl} \triangleleft(x) & = & 1 \\ \triangleright(x) & = & x \end{array}$$

Example run:

$$\triangleleft\mathtt{t}\triangleright \qquad\qquad (1_0 1_2 0_1 = \mathbf{3})$$

Mixed-base encoding

System $\mathcal{M}$:

$$
\begin{array}{llll}
\mathtt{f}\triangleright \to \triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \to \mathtt{2}\triangleright & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}
$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$
\begin{aligned}
\mathtt{f}(x) &= 2x \\
\mathtt{t}(x) &= 2x+1
\end{aligned}
\qquad
\begin{aligned}
\mathtt{0}(x) &= 3x \\
\mathtt{1}(x) &= 3x+1 \\
\mathtt{2}(x) &= 3x+2
\end{aligned}
\qquad
\begin{aligned}
\triangleleft(x) &= 1 \\
\triangleright(x) &= x
\end{aligned}
$$

Example run:

$$\triangleleft 2 \triangleright \qquad (1_0 2_3 0_1 = \mathbf{5})$$

Mixed-base encoding

System $\mathcal{M}$:

$$
\begin{array}{llll}
\texttt{f}\triangleright \to \triangleright & \texttt{f0} \to \texttt{0f} & \texttt{t0} \to \texttt{1t} & \triangleleft\texttt{0} \to \triangleleft\texttt{t} \\
\texttt{t}\triangleright \to \texttt{2}\triangleright & \texttt{f1} \to \texttt{0t} & \texttt{t1} \to \texttt{2f} & \triangleleft\texttt{1} \to \triangleleft\texttt{ff} \\
 & \texttt{f2} \to \texttt{1f} & \texttt{t2} \to \texttt{2t} & \triangleleft\texttt{2} \to \triangleleft\texttt{ft}
\end{array}
$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$
\begin{array}{lll}
\begin{aligned} \texttt{f}(x) &= 2x \\ \texttt{t}(x) &= 2x+1 \end{aligned} &
\begin{aligned} \texttt{0}(x) &= 3x \\ \texttt{1}(x) &= 3x+1 \\ \texttt{2}(x) &= 3x+2 \end{aligned} &
\begin{aligned} \triangleleft(x) &= 1 \\ \triangleright(x) &= x \end{aligned}
\end{array}
$$

Example run:

$$\triangleleft\texttt{ft}\triangleright \qquad (1_0 0_2 1_2 0_1 = \mathbf{5})$$

Mixed-base encoding

System $\mathcal{M}$:

$$
\begin{array}{llll}
\mathtt{f}\triangleright \to \triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \to \mathtt{2}\triangleright & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}
$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$
\begin{aligned}
\mathtt{f}(x) &= 2x \\
\mathtt{t}(x) &= 2x+1
\end{aligned}
\qquad
\begin{aligned}
\mathtt{0}(x) &= 3x \\
\mathtt{1}(x) &= 3x+1 \\
\mathtt{2}(x) &= 3x+2
\end{aligned}
\qquad
\begin{aligned}
\triangleleft(x) &= 1 \\
\triangleright(x) &= x
\end{aligned}
$$

Example run:

$$\triangleleft\mathtt{f2}\triangleright \qquad (1_0 0_2 2_3 0_1 = \mathbf{8})$$

Mixed-base encoding

System $\mathcal{M}$:

$$
\begin{array}{llll}
\mathtt{f}\triangleright \to \triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \to \mathtt{2}\triangleright & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}
$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$
\begin{aligned} \mathtt{f}(x) &= 2x \\ \mathtt{t}(x) &= 2x+1 \end{aligned}
\qquad
\begin{aligned} \mathtt{0}(x) &= 3x \\ \mathtt{1}(x) &= 3x+1 \\ \mathtt{2}(x) &= 3x+2 \end{aligned}
\qquad
\begin{aligned} \triangleleft(x) &= 1 \\ \triangleright(x) &= x \end{aligned}
$$

Example run:

$$\triangleleft\mathtt{1f}\triangleright \qquad (1_0 1_3 0_2 0_1 = \mathbf{8})$$

Mixed-base encoding

System $\mathcal{M}$:

$$
\begin{array}{llll}
\mathtt{f}\triangleright \to \triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \to \mathtt{2}\triangleright & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}
$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$
\begin{aligned}
\mathtt{f}(x) &= 2x \\
\mathtt{t}(x) &= 2x+1
\end{aligned}
\qquad
\begin{aligned}
\mathtt{0}(x) &= 3x \\
\mathtt{1}(x) &= 3x+1 \\
\mathtt{2}(x) &= 3x+2
\end{aligned}
\qquad
\begin{aligned}
\triangleleft(x) &= 1 \\
\triangleright(x) &= x
\end{aligned}
$$

Example run:

$$\triangleleft\mathtt{fff}\triangleright \qquad (1_0 0_2 0_2 0_2 0_1 = \mathbf{8})$$

Mixed-base encoding

System $\mathcal{M}$:

$$
\begin{array}{llll}
\mathtt{f}\triangleright \to \triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \to \mathtt{2}\triangleright & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}
$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$
\begin{aligned}
\mathtt{f}(x) &= 2x \\
\mathtt{t}(x) &= 2x + 1
\end{aligned}
\qquad
\begin{aligned}
\mathtt{0}(x) &= 3x \\
\mathtt{1}(x) &= 3x + 1 \\
\mathtt{2}(x) &= 3x + 2
\end{aligned}
\qquad
\begin{aligned}
\triangleleft(x) &= 1 \\
\triangleright(x) &= x
\end{aligned}
$$

Example run:

$$\triangleleft\mathtt{ff}\triangleright \qquad (1_0 0_2 0_2 0_1 = \mathbf{4})$$

Mixed-base encoding

System $\mathcal{M}$:

$$\begin{array}{llll}
\mathtt{f}\triangleright \to \triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \to \mathtt{2}\triangleright & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$\begin{aligned} \mathtt{f}(x) &= 2x \\ \mathtt{t}(x) &= 2x+1 \end{aligned} \qquad \begin{aligned} \mathtt{0}(x) &= 3x \\ \mathtt{1}(x) &= 3x+1 \\ \mathtt{2}(x) &= 3x+2 \end{aligned} \qquad \begin{aligned} \triangleleft(x) &= 1 \\ \triangleright(x) &= x \end{aligned}$$

Example run:

$$\triangleleft\mathtt{f}\triangleright \qquad (1_0 0_2 0_1 = 2)$$

Mixed-base encoding

System $\mathcal{M}$:

$$
\begin{array}{llll}
\mathtt{f}\triangleright \to \triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \to \mathtt{2}\triangleright & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}
$$

Digits: (Think of the digit n_b as applying the function $bx + n$.)

$$
\begin{aligned}
\mathtt{f}(x) &= 2x \\
\mathtt{t}(x) &= 2x+1
\end{aligned}
\qquad
\begin{aligned}
\mathtt{0}(x) &= 3x \\
\mathtt{1}(x) &= 3x+1 \\
\mathtt{2}(x) &= 3x+2
\end{aligned}
\qquad
\begin{aligned}
\triangleleft(x) &= 1 \\
\triangleright(x) &= x
\end{aligned}
$$

Example run:

$$\triangleleft\triangleright \qquad (1_0 0_1 = \mathbf{1})$$

Nonexistence of matrix proofs for unary encoding

Definition

A sequence $x_1, x_2, \ldots$ is $\mathbb{N}$-*rational* if for some $d \in \mathbb{N}^+$ there exist $\mathbf{M} \in \mathbb{N}^{d \times d}$ and $\mathbf{v}, \mathbf{w} \in \mathbb{N}^d$ such that $x_n = \mathbf{v}^\intercal \mathbf{M}^n \mathbf{w}$ for all n.

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Lemma (A "corollary" of Berstel's theorem (1971))

No $\mathbb{N}$-rational sequence $x_1, x_2, \ldots$ satisfies $x_{8n+1} > x_{9n+2}$ for all n.

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No $\mathbb{N}$-rational sequence $x_1, x_2, \ldots$ satisfies $x_{8n+1} > x_{9n+2}$ for all n.

Lemma (Endrullis, Waldmann, Zantema (2008))

If some collection of d-dimensional affine interpretations is useful for proving the termination of a rewriting system, then so is some other collection of $(d+1)$-dimensional linear interpretations.

Nonexistence of matrix proofs for unary encoding (cont.)

Theorem

Natural matrix interpretations (regardless of dimension) cannot be used for proving the termination of the rewriting system $\mathcal{U}$.

Proof idea. Suppose for a contradiction that there is a collection $\{\mathbf{M}_s\}_{s \in \Sigma}$ of (WLOG linear) d-dimensional interpretations for proving the termination of $\mathcal{U}$.

Nonexistence of matrix proofs for unary encoding (cont.)

Theorem

Natural matrix interpretations (regardless of dimension) cannot be used for proving the termination of the rewriting system $\mathcal{U}$.

Proof idea. Suppose for a contradiction that there is a collection $\{\mathbf{M}_s\}_{s\in\Sigma}$ of (WLOG linear) d-dimensional interpretations for proving the termination of $\mathcal{U}$.

In this system, we may represent a positive integer k as $\diamond\mathtt{h}\mathtt{1}^k\diamond$. Define

$$f_k := \mathbf{e}_1^\mathsf{T}[\diamond\mathtt{h}\mathtt{1}^k\diamond](\mathbf{e}_d) = \mathbf{e}_1^\mathsf{T}\mathbf{M}_{\diamond\mathtt{h}\mathtt{1}^k\diamond}\mathbf{e}_d = (\mathbf{e}_1^\mathsf{T}\mathbf{M}_{\diamond\mathtt{h}})\mathbf{M}_1^k(\mathbf{M}_\diamond\mathbf{e}_d),$$

so $f_1, f_2, \ldots$ is an $\mathbb{N}$-rational sequence.

Nonexistence of matrix proofs for unary encoding (cont.)

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Natural matrix interpretations (regardless of dimension) cannot be used for proving the termination of the rewriting system $\mathcal{U}$.

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so $f_1, f_2, \ldots$ is an $\mathbb{N}$-rational sequence.

Applying the Collatz map three times to a number of the form $8n+1$ gives $8n+1 \mapsto 12n+2 \mapsto 6n+1 \mapsto 9n+2$. Thus, $\mathcal{U}$ allows the rewrite sequence

$$\diamond\mathtt{h}\mathtt{1}^{8n+1}\diamond \quad\to\quad \cdots \quad\to\quad \diamond\mathtt{h}\mathtt{1}^{9n+2}\diamond,$$

which requires the use of every single rule in the system.

Nonexistence of matrix proofs for unary encoding (cont.)

Theorem

Natural matrix interpretations (regardless of dimension) cannot be used for proving the termination of the rewriting system $\mathcal{U}$.

Proof idea. Suppose for a contradiction that there is a collection $\{\mathbf{M}_s\}_{s\in\Sigma}$ of (WLOG linear) d-dimensional interpretations for proving the termination of $\mathcal{U}$.

In this system, we may represent a positive integer k as $\diamond\mathtt{h}\mathtt{1}^k\diamond$. Define

$$f_k := \mathbf{e}_1^\mathsf{T}[\diamond\mathtt{h}\mathtt{1}^k\diamond](\mathbf{e}_d) = \mathbf{e}_1^\mathsf{T}\mathbf{M}_{\diamond\mathtt{h}\mathtt{1}^k\diamond}\mathbf{e}_d = (\mathbf{e}_1^\mathsf{T}\mathbf{M}_{\diamond\mathtt{h}})\mathbf{M}_\mathtt{1}^k(\mathbf{M}_\diamond\mathbf{e}_d),$$

so $f_1, f_2, \ldots$ is an $\mathbb{N}$-rational sequence.

Applying the Collatz map three times to a number of the form $8n+1$ gives $8n+1 \mapsto 12n+2 \mapsto 6n+1 \mapsto 9n+2$. Thus, $\mathcal{U}$ allows the rewrite sequence

$$\diamond\mathtt{h}\mathtt{1}^{8n+1}\diamond \quad \to \quad \cdots \quad \to \quad \diamond\mathtt{h}\mathtt{1}^{9n+2}\diamond,$$

which requires the use of every single rule in the system.

Since $\{\mathbf{M}_s\}_{s\in\Sigma}$ is useful, we have $f_{8n+1} > f_{9n+2}$, contradicting earlier lemma. $\square$

Even worse

Zantema's challenge (2003):

$$W(n) = \begin{cases} 3n/2 & \text{if } n \equiv 0 \pmod 2 \\ \perp & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Unary system:

$$\begin{aligned} \mathtt{h11} &\rightarrow \mathtt{1h} \\ \mathtt{1h}\diamond &\rightarrow \mathtt{1t}\diamond \\ \mathtt{1t} &\rightarrow \mathtt{t111} \\ \diamond\mathtt{t} &\rightarrow \diamond\mathtt{h} \end{aligned}$$

Switching to mixed-base encoding

Zantema's challenge (2003):

$$W(n) = \begin{cases} 3n/2 & \text{if } n \equiv 0 \pmod 2 \\ \perp & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Mixed-base system:

$$\begin{array}{llll}
\mathtt{f}\triangleright \to \mathtt{0}\triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
 & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}$$

Switching to mixed-base encoding

Mixed-base system:

$$
\begin{array}{llll}
\mathtt{f}\triangleright \to \mathtt{0}\triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
 & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}
$$

Interpretations:

$$[\mathtt{f}](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [\mathtt{t}](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[\triangleleft](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x} \qquad [\triangleright](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x}$$

$$[\mathtt{0}](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad [\mathtt{1}](\mathsf{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad [\mathtt{2}](\mathsf{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Switching to mixed-base encoding

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$$[\mathtt{f}](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [\mathtt{t}](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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Example run:

$$[\langle 3360 \rangle](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Switching to mixed-base encoding

Interpretations:

$$[\mathtt{f}](\mathsf{x}) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}1\\1\end{bmatrix} \qquad [\mathtt{t}](\mathsf{x}) = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}0\\1\end{bmatrix}$$

$$[\triangleleft](\mathsf{x}) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\mathsf{x} \qquad [\triangleright](\mathsf{x}) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\mathsf{x}$$

$$[\mathtt{0}](\mathsf{x}) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}0\\2\end{bmatrix} \qquad [\mathtt{1}](\mathsf{x}) = \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}2\\2\end{bmatrix} \qquad [\mathtt{2}](\mathsf{x}) = \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}2\\2\end{bmatrix}$$

Example run:

$$[\langle 2^5 \cdot 3 \cdot 5 \cdot 7 \rangle](\mathsf{x}) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\mathsf{x} + \begin{bmatrix}5\\0\end{bmatrix}$$

Switching to mixed-base encoding

Interpretations:

$$[\mathtt{f}](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [\mathtt{t}](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[\triangleleft](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x} \qquad [\triangleright](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x}$$

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Example run:

$$[\langle 3360 \rangle](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

Switching to mixed-base encoding

Interpretations:

$$[\mathtt{f}](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [\mathtt{t}](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[\triangleleft](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x} \qquad [\triangleright](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x}$$

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Example run:

$$[\langle 5040 \rangle](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

Switching to mixed-base encoding

Interpretations:

$$[\mathtt{f}](\mathsf{x}) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}1\\1\end{bmatrix} \qquad [\mathtt{t}](\mathsf{x}) = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}0\\1\end{bmatrix}$$

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Example run:

$$[\langle 7560 \rangle](\mathsf{x}) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\mathsf{x} + \begin{bmatrix}3\\0\end{bmatrix}$$

Switching to mixed-base encoding

Interpretations:

$$[\mathtt{f}](\mathsf{x}) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}1\\1\end{bmatrix} \qquad [\mathtt{t}](\mathsf{x}) = \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}0\\1\end{bmatrix}$$

$$[\triangleleft](\mathsf{x}) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\mathsf{x} \qquad [\triangleright](\mathsf{x}) = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\mathsf{x}$$

$$[\mathtt{0}](\mathsf{x}) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}0\\2\end{bmatrix} \qquad [\mathtt{1}](\mathsf{x}) = \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}2\\2\end{bmatrix} \qquad [\mathtt{2}](\mathsf{x}) = \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\mathsf{x} + \begin{bmatrix}2\\2\end{bmatrix}$$

Example run:

$$[\langle 11340 \rangle](\mathsf{x}) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\mathsf{x} + \begin{bmatrix}2\\0\end{bmatrix}$$

Switching to mixed-base encoding

Interpretations:

$$[\mathtt{f}](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [\mathtt{t}](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[\triangleleft](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x} \qquad [\triangleright](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x}$$

$$[\mathtt{0}](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad [\mathtt{1}](\mathsf{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad [\mathtt{2}](\mathsf{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

Example run:

$$[\langle \mathbf{17010} \rangle](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Switching to mixed-base encoding

Interpretations:

$$[\mathtt{f}](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad [\mathtt{t}](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[\triangleleft](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x} \qquad [\triangleright](\mathsf{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x}$$

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Example run:

$$[\langle 25515 \rangle](\mathsf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Switching to mixed-base encoding (cont.)

Farkas' map (2005):

$$F(n) = \begin{cases} \frac{n-1}{3} & \text{if } n \equiv 1 \pmod 3 \\ \frac{n}{2} & \text{if } n \equiv 0 \text{ or } n \equiv 2 \pmod 6 \\ \frac{3n+1}{2} & \text{if } n \equiv 3 \text{ or } n \equiv 5 \pmod 6 \end{cases}$$

Mixed-base system:

$$\begin{array}{rclcrclcrclcrcl}
\mathtt{1}\triangleright & \to & \triangleright & & \mathtt{f0} & \to & \mathtt{0f} & & \mathtt{t0} & \to & \mathtt{1t} & & \triangleleft\mathtt{0} & \to & \triangleleft\mathtt{t} \\
\mathtt{0f}\triangleright & \to & \mathtt{0}\triangleright & & \mathtt{f1} & \to & \mathtt{0t} & & \mathtt{t1} & \to & \mathtt{2f} & & \triangleleft\mathtt{1} & \to & \triangleleft\mathtt{ff} \\
\mathtt{1f}\triangleright & \to & \mathtt{1}\triangleright & & \mathtt{f2} & \to & \mathtt{1f} & & \mathtt{t2} & \to & \mathtt{2t} & & \triangleleft\mathtt{2} & \to & \triangleleft\mathtt{ft} \\
\mathtt{1t}\triangleright & \to & \mathtt{12}\triangleright & & & & & & & & & & & & \\
\mathtt{2t}\triangleright & \to & \mathtt{22}\triangleright & & & & & & & & & & & &
\end{array}$$

Switching to mixed-base encoding (cont.)

Interpretations:

$$[\mathtt{f}](\mathsf{x}) = \begin{bmatrix} - & - & - & 2 & - \\ - & 2 & 0 & - & - \\ 2 & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} \mathsf{x} \oplus \begin{bmatrix} 0 \\ - \\ - \\ - \\ - \end{bmatrix} \qquad [\mathtt{t}](\mathsf{x}) = \begin{bmatrix} - & - & - & - & 2 \\ 0 & 2 & 0 & - & 0 \\ 2 & - & 2 & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} \mathsf{x} \oplus \begin{bmatrix} 0 \\ - \\ - \\ - \\ - \end{bmatrix}$$

$$[\triangleleft](\mathsf{x}) = \begin{bmatrix} 0 \\ 2 \\ - \\ - \\ 4 \end{bmatrix} \qquad [\triangleright](\mathsf{x}) = \begin{bmatrix} 0 & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} \mathsf{x}$$

$$[0](\mathsf{x}) = \begin{bmatrix} 0 & 4 & 0 & - & - \\ - & 4 & - & - & - \\ - & 4 & 0 & - & - \\ 0 & 3 & 0 & - & - \\ - & - & - & - & - \end{bmatrix} \mathsf{x}$$

$$[1](\mathsf{x}) = \begin{bmatrix} 1 & - & - & - & - \\ - & 4 & 0 & - & - \\ - & 4 & 0 & - & - \\ 0 & - & - & - & - \\ 0 & 3 & 0 & - & - \end{bmatrix} \mathsf{x}$$

$$[2](\mathsf{x}) = \begin{bmatrix} 0 & - & 0 & - & - \\ - & 4 & - & - & - \\ 0 & - & 1 & - & 0 \\ - & - & - & - & - \\ 0 & - & 0 & - & 0 \end{bmatrix} \mathsf{x}$$

Theorem. *For all $n \in \mathbb{N}^+$, the trajectory $F^{\mathbb{N}}(n)$ contains 1.*

Proof. We proceed by induction. For $n = 1$ the result holds since $F^{\mathbb{N}}(1) = (1, 0, 0, \dots)$. Assuming it holds for all positive integers less than n, we will show that it holds for n. In particular, we will show that the trajectory of n reaches a number strictly smaller than n, which implies by the induction hypothesis that the trajectory contains 1. We split into three cases.

(i) Assume $n \equiv 1 \pmod 3$. Then $F(n) = (n-1)/3$ is smaller than n.

(ii) Assume $n \equiv 0$ or $n \equiv 2 \pmod 6$. Then $F(n) = n/2$ is smaller than n.

(iii) Assume $n \equiv 3$ or $n \equiv 5 \pmod 6$. Denote n by N_1, and consider the trajectory $F^{\mathbb{N}}(N_1) = (N_1, N_2, N_3, \dots)$. Let

$$k = \sup \left\{ i \in \mathbb{N}^+ \;\middle|\; N_j = \frac{3N_{j-1}+1}{2} \text{ for all } 1 < j \le i \right\}.$$

We cannot have $k = \infty$, i.e., F cannot apply $N \mapsto (3N+1)/2$ indefinitely, since for such a sequence we have

$$N_j + 1 = \left(\frac{3}{2}\right)(N_{j-1}+1) = \left(\frac{3}{2}\right)^{j-1}(N_1+1).$$

As all the elements in the trajectory have to be integers, $k-1$ cannot exceed the number of times that N_1+1 can be divided by 2, so k is finite. Consider

$$N_{k+1} = \left(\frac{3}{2}\right)^k (N_1+1) - 1.$$

It is of the form $3^k \cdot L - 1$ for some positive integer L, so we know that either $N_{k+1} \equiv 2 \pmod 6$ or $N_{k+1} \equiv 5 \pmod 6$. Due to the way k is defined, N_{k+1} cannot be congruent to $5 \pmod 6$, so we have $N_{k+1} \equiv 2 \pmod 6$. In particular, N_{k+1} is even, so $\left(\frac{3}{2}\right)^k (N_1+1)$ is odd, which in turn implies that $\frac{N_1+1}{2^k}$ is odd, i.e., we have $N_1 = 2^k \cdot (2M+1) - 1$ for some natural number M. Then, N_{k+1} satisfies

$$N_{k+1} = 3^k \cdot (2M+1) - 1 = 3^k \cdot 2M + 3^k - 1.$$

Since $N_{k+1} \equiv 2 \pmod 6$, we can deduce that

$$F(N_{k+1}) = \frac{N_{k+1}}{2} = \frac{3^k \cdot 2M + 3^k - 1}{2} \equiv 1 \pmod 3,$$

and furthermore,

$$F^2(N_{k+1}) = \frac{\frac{3^k \cdot 2M + 3^k - 1}{2} - 1}{3} = 3^{k-1} \cdot 2M + 3^{k-1} - 1 \equiv 2 \pmod 6.$$

Repeatedly applying F in the above manner gives $F^{2k}(N_{k+1}) = 2M$. Since $k \ge 1$, we have $2M < 2^k \cdot (2M+1) - 1 = N_1 = n$, and so the trajectory $F^{\mathbb{N}}(n)$ reaches a number strictly smaller than n. $\square$