Exponential separations using guarded extension variables

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Resolution

Refutes a propositional formula in conjunctive normal form (i.e., a set of clauses) by using the single rule

$$\frac{A \vee x \qquad B \vee \neg x}{A \vee B}$$

to derive the empty clause, which is trivially false.

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Throughout this talk, "proof" means "refutation" since

proof of unsatisfiability $\equiv$ refutation of satisfiability.

Example: resolution proof

$$\Gamma = (\overline{x} \vee \overline{z}) \wedge (\overline{y} \vee z) \wedge (x \vee y \vee \overline{z}) \wedge (x \vee \overline{y}) \wedge (y \vee z)$$

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Tree-like:

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Sequence-like:

$$\overline{x} \vee \overline{z},\ \overline{y} \vee z,\ x \vee y \vee \overline{z},\ x \vee \overline{y},\ y \vee z,\ z,\ x \vee y,\ x,\ \overline{x},\ \bot$$

Proof complexity

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Let P and Q be proof systems.

- P *simulates* Q if there exists some c such that $\mathsf{s}_P(\Gamma) \leq \mathsf{s}_Q(\Gamma)^c$ for all Γ.
- P *is separated from* Q if there exists some sequence $(\Gamma_n)_{n\in\mathbb{N}}$ such that $\mathsf{s}_P(\Gamma_n) = n^{O(1)}$ while $\mathsf{s}_Q(\Gamma_n) = 2^{\Omega(n)}$.

More examples of proof systems

resolution, DNF-resolution, sequent calculus,
Frege, extended Frege, substitution Frege, AC^0*-Frege,*
cutting planes, Nullstellensatz, polynomial calculus,
Sherali–Adams, Lasserre, Lovász–Schrijver, . . .

They capture various restricted forms of mathematical reasoning: logical, geometric, algebraic, etc.

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This talk: Proof systems that capture reasoning WLOG

Example: deriving clauses WLOG

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Note that x is not necessarily logically implied by Γ:

There might be an assignment satisfying Γ *without setting* $x = 1$.

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Lemma

A clause C is redundant with respect to a formula Γ if and only if there exists a partial assignment τ such that

$$\Gamma \wedge \neg C \models (\Gamma \wedge C)|_\tau.$$

Sufficient syntactic criteria for redundancy

Definition (Blocked clause)

A clause $C = x \vee C'$ is *blocked* for x with respect to a formula Γ if, for every clause D of the form $\neg x \vee D'$ in Γ,

$$C' \vee D' \text{ is tautological.}$$

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Example (Blocked clause)

$$C = x \vee y \vee \neg z$$
$$\Gamma = (\neg x \vee \neg y) \wedge (\neg x \vee z) \wedge (y \vee z)$$

C is blocked for x with respect to Γ.

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Why? Consider some total assignment α that satisfies $\Gamma \wedge \neg C$.
$\Gamma \wedge C$ is satisfied by $\alpha \circ \tau$, which is α with $\alpha(x)$ flipped.

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What about $\neg x \vee D'$? There is some $y \in C'$ such that $\neg y \in D'$.
$\alpha \circ \tau$ still sets y to $\alpha(y) = 0$.

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Definition (Resolution asymmetric tautology*)

A clause $C = x \vee C'$ is a *RAT* for x with respect to a formula Γ if, for every clause D of the form $\neg x \vee D'$ in Γ,

$$C' \vee D' \text{ is tautological or subsumed by } \Gamma.$$

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Definition (Set-blocked clause)

A clause $C = L \vee C'$ is an *SBC* for L with respect to a formula Γ if, for every clause D in Γ such that $D \cap \neg L \neq \varnothing$ and $D \cap L = \varnothing$,

$$(C \setminus L) \vee (D \setminus \neg L) \text{ is tautological.}$$

Inference rules

Definition

A *proof* in a proof system P of a formula Γ is a sequence $\Gamma_1, \ldots, \Gamma_N$ of formulas such that

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- Γ_{i+1} follows from Γ_i by some rule of P.

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- *Redundancy:* Add to Γ_i a syntactically redundant clause.
- *Deletion:* Remove a clause from Γ_i.

Proof systems

- ER — resolution + extension
- BC — resolution + blocked clause addition
- BC^{-} — BC without new variables
- DBC — BC with deletion

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- SPR — resolution + "SBC $\times$ RAT" addition

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"Fact"

BC generalizes ER $\implies$ BC^- is more interesting

Thus, we focus on the versions without new variables.

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Definition (Generalized extended resolution)

A GER proof of a formula Γ is a BC proof of some subset $\Gamma' \subseteq \Gamma$ such that every clause from $\Gamma \setminus \Gamma'$ is derived in the proof as blocked.

Some known results for generalizations of BC^-

Upper bounds

- Pigeonhole principle
- Bit pigeonhole principle
- Parity principle
- Clique-coloring principle
- Tseitin tautologies
- OR-ification, XOR-ification, lifting with index gadgets

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Lower bounds

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Simulations

- Deletion collapses the systems
- No known simulations without deletion (except obvious ones)

In a *relaxed* sense, BC^- simulates ER

(without new variables!)

Lemma

Suppose that a formula Γ has an ER proof of size m and that $X = (y \vee x_1 \vee \cdots \vee x_m) \wedge y$ has no variables in common with Γ. Then $\Gamma \wedge X$ has a BC^- proof of size $O(m)$.

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Proof. Consider a use of the extension rule in the ER proof that introduces $x_i \leftrightarrow p \wedge q$. WLOG, the literals p and q are not new.

Add the clauses

$$\neg x_i \vee \neg y \vee p \qquad \neg x_i \vee \neg y \vee q \qquad x_i \vee \neg p \vee \neg q$$

in sequence as blocked clauses. Resolve against y. $\square$

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The idea of this result will be useful in proving separations across various generalizations of BC^-.

Moreover, the proofs of the separations will be completely modular, using previously known upper and lower bounds as black boxes.

Guarded extension variables

Let Γ be a formula with an ER proof of size $m = |\Gamma|^{O(1)}$.
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P will somehow use the included variables to simulate the ER proof.
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Example

With respect to the formula $(\neg x \vee y) \wedge (\neg x \vee \neg y)$, any clause blocked for x has to include both $\neg y$ and y.

Example: lower bound

Lemma

$f(\Gamma) \coloneqq \Gamma \wedge \bigwedge_{i=1}^{m} \left[(x_i \vee \Gamma) \wedge (\neg x_i \vee \Gamma) \right]$ is no easier than Γ for BC^{-}.

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3. Since $\Gamma \subseteq f(\Gamma)$, every clause in Δ is in particular blocked wrt Γ.
4. For the assignment $\alpha(x_i) = 1$, we have $(f(\Gamma) \wedge \Delta)|_\alpha = \Gamma \wedge \Delta'$, where Δ' is possible to derive from Γ in BC^-.

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5. Resolution is closed under restrictions, which implies that $f(\Gamma) \wedge \Delta$ is at least as hard for resolution as $\Gamma \wedge \Delta'$.

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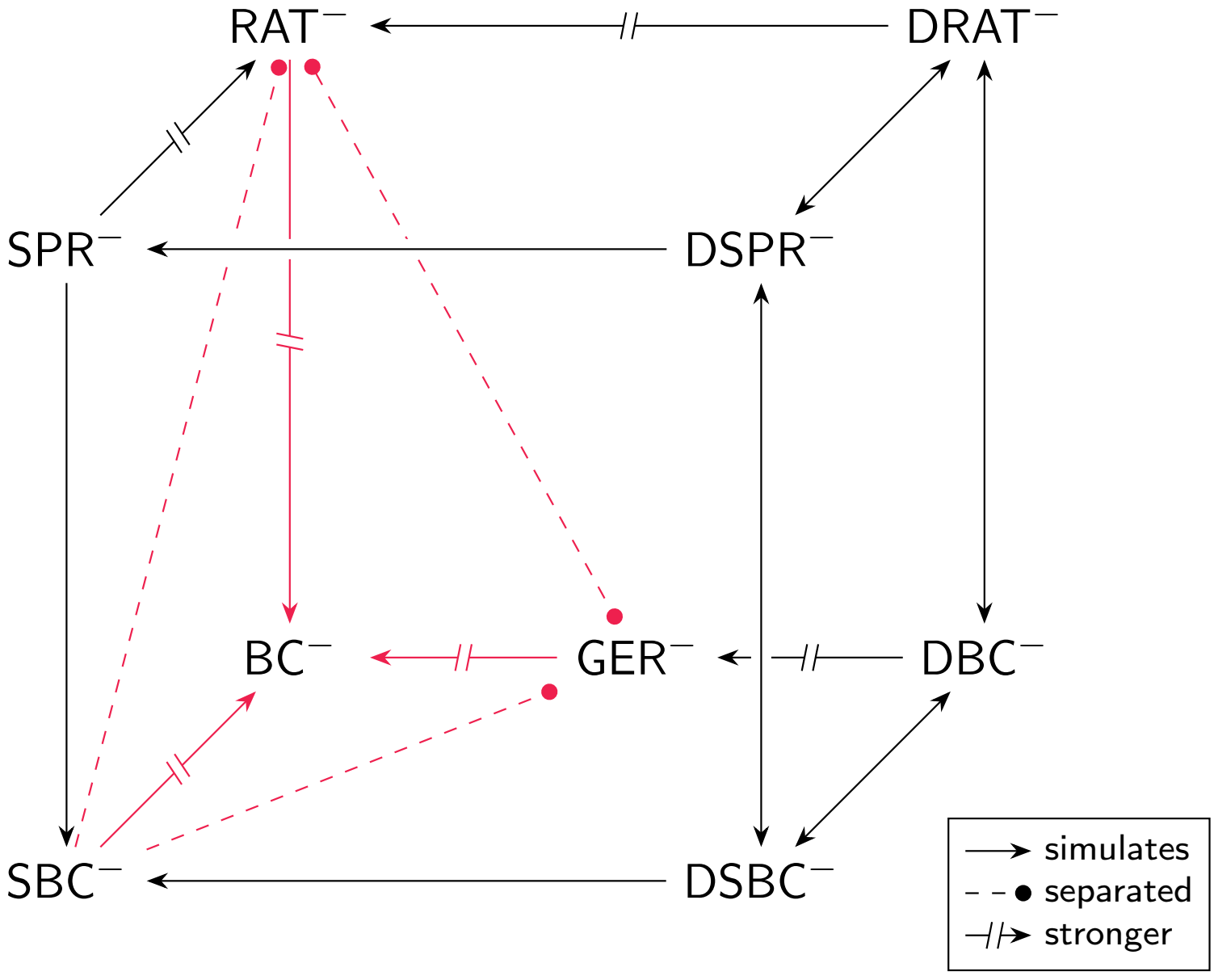
5. Resolution is closed under restrictions, which implies that $f(\Gamma) \wedge \Delta$ is at least as hard for resolution as $\Gamma \wedge \Delta'$.
6. If $\Gamma \wedge \Delta'$ is easy for resolution, then Γ is easy for BC^-. $\square$

Summary of the new results



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$\mathrm{SBC}^- \not\geq \mathrm{RAT}^-$ (*)

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$f(\Gamma)$ is easy for RAT^- (regardless of whether Γ is).

$f(\Gamma)$ is at least as hard as Γ for GER^- and SBC^-.

Separating constructions

Let Γ be a formula with an ER proof of size $m = |\Gamma|^{O(1)}$.

$\mathrm{RAT}^- \not\geq \mathrm{GER}^-$
$\mathrm{RAT}^- \not\geq \mathrm{SBC}^-$

$$g(\Gamma) := \Gamma \wedge \bigwedge_{i=1}^{m} \left[(\neg x_i \vee y_i) \wedge (x_i \vee \neg y_i) \right]$$

$g(\Gamma)$ is easy for both GER^- and SBC^- (for different reasons).

$g(\Gamma)$ is at least as hard as Γ for RAT^-.

Separating constructions

Let Γ be a formula with an ER proof of size $m = |\Gamma|^{O(1)}$.

$\mathrm{SBC}^- \not\geq \mathrm{GER}^-$ (*)

$$h_s(\Gamma) := \Gamma \wedge \bigwedge_{i=1}^{m} \bigwedge_{j=1}^{s} \left[(x_i \vee y_j \vee \neg z_j) \wedge (\neg x_i \vee y_j \vee \neg z_j) \right]$$
$$\wedge \bigwedge_{j=1}^{s} \left[(\neg y_j \vee z_j) \wedge (y_j \vee \Gamma) \wedge (\neg z_j \vee \Gamma) \right]$$

$h_s(\Gamma)$ is easy for GER^- (regardless of whether Γ is).

$h_s(\Gamma)$ is at least as hard as Γ for SBC^- with suitable choice of s.

Take home

Results

- Recipe for proving separations in a modular way
- Almost complete picture of the weakest generalizations of BC^{-}

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- Almost complete picture of the weakest generalizations of BC^-

Open questions

- Lower bounds for SBC^- or SPR^-
- Separations using natural combinatorial principles
- Any subsystem of Frege above resolution that DBC^- simulates
- Other uses of the high-level idea in proof complexity

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