Regular resolution effectively simulates resolution

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Resolution [Blake, 1937; Robinson, 1965]

Refutes a propositional formula in conjunctive normal form (i.e., a set of clauses) by using the single rule

$$\frac{A \vee x \qquad B \vee \overline{x}}{A \vee B}$$

to derive the empty clause, which is trivially false.

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Throughout this talk, "proof" means "refutation":

$$\text{proof of unsatisfiability} \quad \equiv \quad \text{refutation of satisfiability}$$

Example: resolution proof

$$\Gamma = (\overline{x} \vee \overline{z}) \wedge (\overline{y} \vee z) \wedge (x \vee y \vee \overline{z}) \wedge (x \vee \overline{y}) \wedge (y \vee z)$$

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Tree-like:

$$\dfrac{\dfrac{x \vee \overline{y} \qquad \dfrac{\dfrac{\overline{y} \vee z \qquad y \vee z}{z} \qquad x \vee y \vee \overline{z}}{x \vee y}}{x} \qquad \dfrac{\dfrac{\overline{y} \vee z \qquad y \vee z}{z} \qquad \overline{x} \vee \overline{z}}{\overline{x}}}{\bot}$$

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Sequence-like:

$$\overline{x} \vee \overline{z},\ \overline{y} \vee z,\ x \vee y \vee \overline{z},\ x \vee \overline{y},\ y \vee z,\ z,\ x \vee y,\ x,\ \overline{x},\ \bot$$

Regular resolution [Tseitin, 1968]

No variable is resolved upon more than once along any path.

$$\dfrac{\dfrac{x \lor \overline{y} \qquad \dfrac{\dfrac{\overline{y} \lor z \qquad y \lor z}{z} \qquad x \lor y \lor \overline{z}}{x \lor y}}{x} \qquad \dfrac{\dfrac{\overline{y} \lor z \qquad y \lor z}{z} \qquad \overline{x} \lor \overline{z}}{\overline{x}}}{\bot}$$

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$$
\dfrac{
  \dfrac{
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    \qquad
    \dfrac{
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      \qquad
      x \lor y \lor \bar{z}
    }{x \lor y}\,z
  }{x}\,y
  \qquad
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Regular resolution is exponentially weaker than resolution.
[Alekhnovich, Johannsen, Pitassi, Urquhart, 2007]

Propositional proof complexity

Concerned with the quantity

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Let P and Q be proof systems.

- *P simulates Q* if there is some c such that $\mathsf{s}_P(\Gamma) \leq \mathsf{s}_Q(\Gamma)^c$ for all Γ.
- *P is exponentially separated from Q* if there is some $(\Gamma_n)_{n\in\mathbb{N}}$ such that $\mathsf{s}_P(\Gamma_n) = n^{O(1)}$ while $\mathsf{s}_Q(\Gamma_n) = 2^{\Omega(n)}$.

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- The formula $f(\Gamma, m)$ is satisfiable if and only if Γ is and it can be computed in time polynomial in $|\Gamma| + m$.
- When m is at least the size of the smallest Q-proof of Γ, the formula $f(\Gamma, m)$ has a P-proof of size polynomial in $|\Gamma| + m$.

Known results

Effective simulations exist in several cases where either no simulation is known or an exponential separation exists.

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- constant-depth extensions of PC $\rightarrow$ $AC^0[p]$-Frege [Impagliazzo, Mouli, Pitassi, 2020]
- G_0 ("quantified Frege") $\rightarrow$ *any* proof system [Pitassi and Santhanam, 2010]

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Relationship between automatizability and effective simulations $\Longrightarrow$

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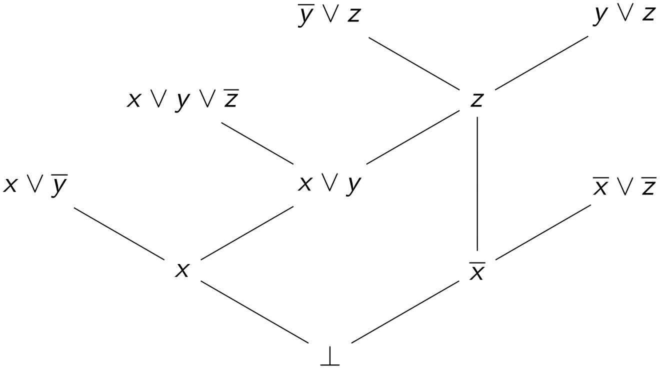
If resolution is not weakly automatizable, then neither is regular resolution.

Proof idea

$$\Gamma = (\overline{x} \vee \overline{z}) \wedge (\overline{y} \vee z) \wedge (x \vee y \vee \overline{z}) \wedge (x \vee \overline{y}) \wedge (y \vee z)$$



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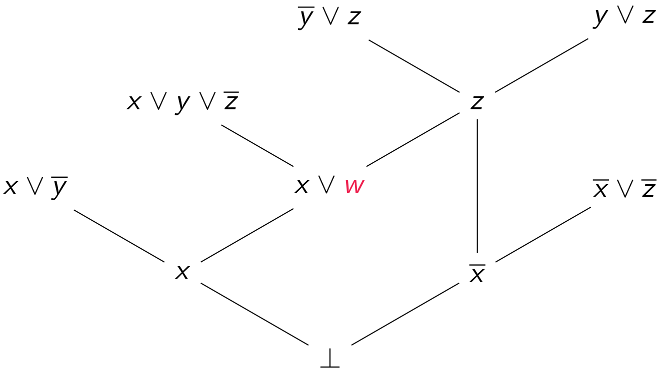
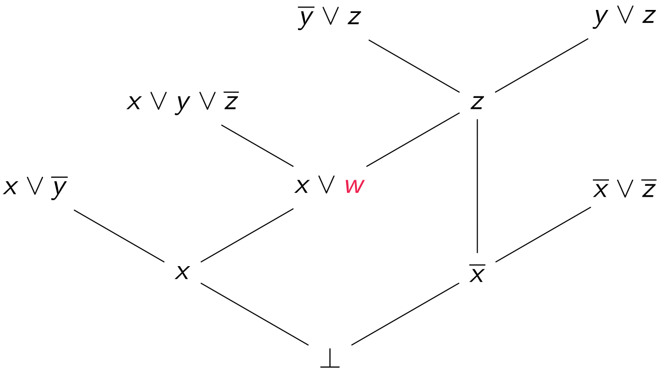


Proof idea

$$f(\Gamma, m) = (y \leftrightarrow w) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{y} \vee z) \wedge (x \vee y \vee \overline{z}) \wedge (x \vee \overline{y}) \wedge (y \vee z)$$



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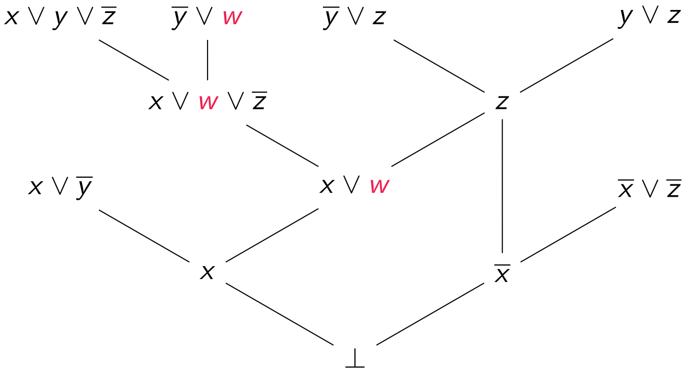

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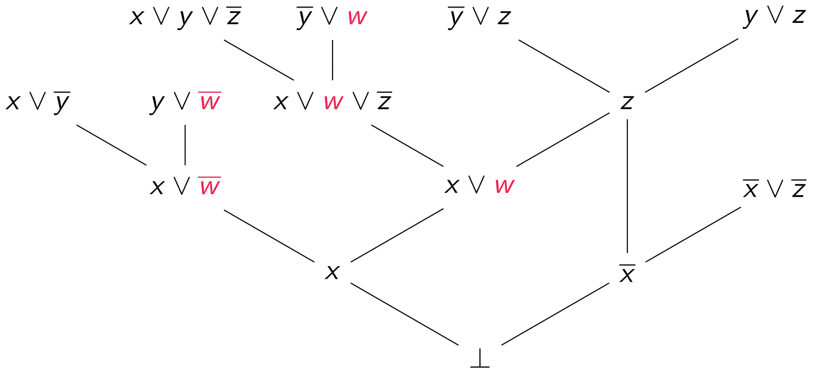

Open questions

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- Use the effective simulation to determine the relationship between regular resolution and linear resolution.
  (This was partially our initial motivation.)
- Are "restarts" necessary for linear resolution and clause learning to simulate resolution?