An Automated Approach to the Collatz Conjecture

(Explorations with Rewriting Systems, Matrix Interpretations, and SAT)

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Automated deduction for mathematics

Examples of conjectures settled with computer assistance:

- Four color theorem (1976)
- Robbins conjecture (1996)
- Kepler conjecture (1998)
- Boolean Pythagorean triples problem (2016)

This work: Approach the Collatz conjecture with the techniques developed for automatically proving termination of string rewriting.

Available at: `https://arxiv.org/abs/2105.14697`

String rewriting

Σ	alphabet
$\ell \to r$	rewrite rule
R	rewriting system
$\to_R$	rewrite relation

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Example: $R = \{ab \to ba\}$

$$\underline{ab}bab \to_R b\underline{ab}ab \to_R bba\underline{ab} \to_R bb\underline{ab}a \to_R bbbaa$$

Termination

R is terminating if there is no infinite chain

$$X_0 \quad \to_R \quad X_1 \quad \to_R \quad X_2 \quad \to_R \quad \cdots$$

where $X_i \in \Sigma^*$.

Termination

Examples:

- $R = \{ab \to ba\}$ terminating
- $S = \{a \to aa\}$ nonterminating
- $T = \{aa \to a\}$ terminating
- $P = \left\{\begin{matrix} a \to b \\ b \to a \end{matrix}\right\}$ nonterminating
- $Z = \left\{\begin{matrix} aa \to bc \\ bb \to ac \\ cc \to ab \end{matrix}\right\}$?
- $G = \left\{\begin{matrix} aaaa \to babb \\ baab \to aaba \end{matrix}\right\}$?

Proving termination [Baader and Nipkow, 1998]

R is terminating if there exists a well-founded order $\succ_{\Sigma^*}$ such that for all $X, Y \in \Sigma^*$

$$X \to_R Y \quad \Longrightarrow \quad X \succ_{\Sigma^*} Y.$$

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Translate $s \in \Sigma$ into an element $[s]$ of another domain $(A, \succ_A)$ and show that $[\ell] \succ_A [r]$ for all $\ell \to r \in R$.

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Examples:

- $T = \{aa \to a\}$. Let $[a] = 1$, extend to strings additively.

$$[aa] = 2 > 1 = [a]$$

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Examples:

- $T = \{aa \to a\}$. Let $[a] = 1$, extend to strings additively.

$$[aa] = 2 > 1 = [a]$$

- $R = \{ab \to ba\}$. Let $[a](x) = x^2$ and $[b](x) = x + 1$, extend to strings compositionally.

$$[ab](x) = (x+1)^2 > x^2 + 1 = [ba](x)$$

Matrix interpretations [Hofbauer and Waldmann, 2006]

Affine functions $[s]\colon \mathbb{N}^d \to \mathbb{N}^d$ (extended to strings compositionally):

$$[s](\mathrm{x}) = \mathrm{M}_s\mathrm{x} + \mathrm{v}_s$$

Well-founded order:

$$\mathrm{x} \succ \mathrm{y} \quad \Longleftrightarrow \quad x_1 > y_1 \ \wedge \ x_i \geq y_i \text{ for } i \in \{2, 3, \ldots, d\}$$

Look for interpretations satisfying

$$[\ell](\mathrm{x}) = \mathrm{M}_\ell\mathrm{x} + \mathrm{v}_\ell \succ \mathrm{M}_r\mathrm{x} + \mathrm{v}_r = [r](\mathrm{x}) \text{ for all } \mathrm{x} \in \mathbb{N}^d.$$

Encode all the resulting constraints as a SAT instance.

Example proof

$$Z = \begin{Bmatrix} aa \to bc \\ bb \to ac \\ cc \to ab \end{Bmatrix}$$

Example proof

$$Z = \left\{ \begin{matrix} aa \to bc \\ bb \to ac \\ cc \to ab \end{matrix} \right\}$$

$$[a](\mathrm{x}) = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix} \mathrm{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$[b](\mathrm{x}) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \mathrm{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$$[c](\mathrm{x}) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathrm{x} + \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Example proof (cont.)

$$\mathsf{x} \succ \mathsf{y} \quad \Longleftrightarrow \quad x_1 > y_1 \ \wedge \ x_i \geq y_i \text{ for } i \in \{2, 3, \ldots, d\}.$$

It is decidable to check the following.

$$[aa](\mathsf{x}) = \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad \succ \quad \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = [bc](\mathsf{x})$$

$$[bb](\mathsf{x}) = \begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 4 \\ 0 \\ 2 \\ 6 \end{bmatrix} \quad \succ \quad \begin{bmatrix} 1 & 1 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 4 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 2 \\ 0 \\ 1 \\ 6 \end{bmatrix} = [ac](\mathsf{x})$$

$$[cc](\mathsf{x}) = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} \quad \succ \quad \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \mathsf{x} + \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} = [ab](\mathsf{x})$$

Collatz conjecture

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \frac{3n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

Conjecture

For all $n \in \mathbb{N}^+$, the trajectory $C(n), C(C(n)), C(C(C(n))), \ldots$ reaches 1.

String rewriting for Collatz [Zantema, 2005]

$$\mathtt{h11} \to \mathtt{1h} \qquad \mathtt{11h}\diamond \to \mathtt{11s}\diamond \qquad \mathtt{h1}\diamond \to \mathtt{t11}\diamond$$

$$\mathtt{1s} \to \mathtt{s1} \qquad \mathtt{1t} \to \mathtt{t111}$$

$$\diamond\mathtt{s} \to \diamond\mathtt{h} \qquad \diamond\mathtt{t} \to \diamond\mathtt{h}$$

Theorem

Above system is terminating $\iff$ *Collatz conjecture holds.*

$$\diamond\mathtt{h}\mathtt{1}^{2n}\diamond \;\to^*\; \diamond\mathtt{h}\mathtt{1}^{n}\diamond \qquad \text{for } n > 1$$

$$\diamond\mathtt{h}\mathtt{1}^{2n+1}\diamond \;\to^*\; \diamond\mathtt{h}\mathtt{1}^{3n+2}\diamond \qquad \text{for } n \geq 0$$

String rewriting for Collatz [Zantema, 2005]

$$\begin{array}{lcl} \mathtt{h11} \to \mathtt{1h} & \quad \mathtt{11h}\diamond \to \mathtt{11s}\diamond & \quad \mathtt{h1}\diamond \to \mathtt{t11}\diamond \\ & \mathtt{1s} \to \mathtt{s1} & \quad \mathtt{1t} \to \mathtt{t111} \\ & \diamond\mathtt{s} \to \diamond\mathtt{h} & \quad \diamond\mathtt{t} \to \diamond\mathtt{h} \end{array}$$

Theorem

There exists no collection of matrix interpretations of any dimension d for the above system such that

- *for some rule $\ell \to r$ we have $[\ell](x) \succ [r](x)$ for all $x \in \mathbb{N}^d$, and*
- *for remaining $\ell' \to r'$ we have $[\ell'](x) \succeq [r'](x)$ for all $x \in \mathbb{N}^d$.*

Trivial test

$$
\begin{aligned}
\texttt{h11} &\rightarrow \texttt{1h} \\
\texttt{1h}\diamond &\rightarrow \texttt{1t}\diamond \\
\texttt{1t} &\rightarrow \texttt{t111} \\
\diamond\texttt{t} &\rightarrow \diamond\texttt{h}
\end{aligned}
$$

Applies $n \mapsto 3n/2$ as long as 2 divides n.

No automated termination proof is known.

Mixed binary–ternary representation

Can view binary representations as compositions of the functions

$$\mathtt{f}(x) = 2x$$
$$\mathtt{t}(x) = 2x + 1$$

along with $\triangleleft(x) = 1$.

Example:

$$12 = (1100)_2 = \mathtt{f}(\mathtt{f}(\mathtt{t}(\triangleleft(x))))$$

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And then: $6/2 = 3 = \quad \mathtt{t}(\triangleleft(x))$

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If we had a function $h(x) = 3x + 1$, we could write

$$10 = h(3) = h(\mathtt{t}(\triangleleft(x))).$$

Mixed binary–ternary representation (cont.)

$$\begin{aligned} \mathtt{f}(x) &= 2x \\ \mathtt{t}(x) &= 2x+1 \end{aligned}$$

$$\begin{aligned} \mathtt{0}(x) &= 3x \\ \mathtt{1}(x) &= 3x+1 \\ \mathtt{2}(x) &= 3x+2 \end{aligned}$$

$$\begin{aligned} \triangleleft(x) &= 1 \\ \triangleright(x) &= x \end{aligned}$$

Mixed binary–ternary representation (cont.)

$$\begin{aligned} \mathtt{f}(x) &= 2x \\ \mathtt{t}(x) &= 2x+1 \end{aligned} \qquad \begin{aligned} \mathtt{0}(x) &= 3x \\ \mathtt{1}(x) &= 3x+1 \\ \mathtt{2}(x) &= 3x+2 \end{aligned} \qquad \begin{aligned} \triangleleft(x) &= 1 \\ \triangleright(x) &= x \end{aligned}$$

Consider $19 = \triangleleft\mathtt{0}\,\mathtt{f}\,\mathtt{1}\triangleright = \triangleright(\mathtt{1}(\mathtt{f}(\mathtt{0}(\triangleleft(x)))))$. Ends with ternary, so we will rewrite.

Mixed binary–ternary representation (cont.)

$$\mathtt{f}(x) = 2x \qquad \mathtt{0}(x) = 3x \qquad \triangleleft(x) = 1$$
$$\mathtt{t}(x) = 2x + 1 \qquad \mathtt{1}(x) = 3x + 1 \qquad \triangleright(x) = x$$
$$\mathtt{2}(x) = 3x + 2$$

Consider $19 = \triangleleft\mathtt{0}\,\mathtt{f}\,\mathtt{1}\triangleright = \triangleright(\mathtt{1}(\mathtt{f}(\mathtt{0}(\triangleleft(x)))))$. Ends with ternary, so we will rewrite.

We have $\mathtt{1}(\mathtt{f}(x)) = 3(2x) + 1 = 6x + 1 = 2(3x) + 1 = \mathtt{t}(\mathtt{0}(x))$.
Could also write $19 = \triangleleft\mathtt{0}\mathtt{0}\mathtt{t}\triangleright$.

Mixed binary–ternary representation (cont.)

$$\begin{aligned} \mathtt{f}(x) &= 2x \\ \mathtt{t}(x) &= 2x+1 \end{aligned} \qquad \begin{aligned} \mathtt{0}(x) &= 3x \\ \mathtt{1}(x) &= 3x+1 \\ \mathtt{2}(x) &= 3x+2 \end{aligned} \qquad \begin{aligned} \lhd(x) &= 1 \\ \rhd(x) &= x \end{aligned}$$

Consider $19 = \lhd\mathtt{0\,f\,1}\rhd = \rhd(\mathtt{1}(\mathtt{f}(\mathtt{0}(\lhd(x)))))$. Ends with ternary, so we will rewrite.

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Could also write $19 = \lhd\mathtt{0\,0\,t}\rhd$.

Apply Collatz map at the rightmost end via

$$C(\rhd(\mathtt{t}(x))) = \frac{3(2x+1)+1}{2} = \frac{6x+4}{2} = 3x+2 = \rhd(\mathtt{2}(x)),$$

obtaining $C(19) = 29 = \lhd\mathtt{0\,0\,2}\rhd$.

Alternative rewriting system for Collatz

$$
\begin{array}{llll}
\mathtt{f}\triangleright \to \triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \to \mathtt{2}\triangleright & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}
$$

Theorem

Above system is terminating $\iff$ *Collatz conjecture holds.*

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$\mathtt{f}\triangleright \to \triangleright$	$\mathtt{f0} \to \mathtt{0f}$	$\mathtt{t0} \to \mathtt{1t}$	$\triangleleft\mathtt{0} \to \triangleleft\mathtt{t}$
$\mathtt{t}\triangleright \to \mathtt{2}\triangleright$	$\mathtt{f1} \to \mathtt{0t}$	$\mathtt{t1} \to \mathtt{2f}$	$\triangleleft\mathtt{1} \to \triangleleft\mathtt{ff}$
	$\mathtt{f2} \to \mathtt{1f}$	$\mathtt{t2} \to \mathtt{2t}$	$\triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}$

Theorem

Above system is terminating $\iff$ *Collatz conjecture holds.*

Example sequence:

12		12		6		6		3		3		5
$\triangleleft\mathtt{f}\underline{\mathtt{f0}}\triangleright$	$\to$	$\triangleleft\mathtt{f0}\underline{\mathtt{f}\triangleright}$	$\to$	$\triangleleft\underline{\mathtt{f0}}\triangleright$	$\to$	$\triangleleft\mathtt{0}\underline{\mathtt{f}\triangleright}$	$\to$	$\underline{\triangleleft\mathtt{0}}\triangleright$	$\to$	$\triangleleft\underline{\mathtt{t}\triangleright}$	$\to$	$\underline{\triangleleft\mathtt{2}}\triangleright$

	5		8		8		8		4		2		1
$\to$	$\triangleleft\mathtt{f}\underline{\mathtt{t}\triangleright}$	$\to$	$\triangleleft\underline{\mathtt{f2}}\triangleright$	$\to$	$\underline{\triangleleft\mathtt{1}}\mathtt{f}\triangleright$	$\to$	$\triangleleft\mathtt{ff}\underline{\mathtt{f}\triangleright}$	$\to$	$\triangleleft\mathtt{f}\underline{\mathtt{f}\triangleright}$	$\to$	$\triangleleft\underline{\mathtt{f}\triangleright}$	$\to$	$\triangleleft\triangleright$

Nontrivial test: Farkas' map [Farkas, 2005]

$$F(n) = \begin{cases} \frac{n-1}{3} & \text{if } n \equiv 1 \pmod 3 \\ \frac{n}{2} & \text{if } n \equiv 0 \text{ or } n \equiv 2 \pmod 6 \\ \frac{3n+1}{2} & \text{if } n \equiv 3 \text{ or } n \equiv 5 \pmod 6 \end{cases}$$

$\mathtt{1}\triangleright \to \triangleright$	$\mathtt{f0} \to \mathtt{0f}$	$\mathtt{t0} \to \mathtt{1t}$	$\triangleleft\mathtt{0} \to \triangleleft\mathtt{t}$
$\mathtt{0f}\triangleright \to \mathtt{0}\triangleright$	$\mathtt{f1} \to \mathtt{0t}$	$\mathtt{t1} \to \mathtt{2f}$	$\triangleleft\mathtt{1} \to \triangleleft\mathtt{ff}$
$\mathtt{1f}\triangleright \to \mathtt{1}\triangleright$	$\mathtt{f2} \to \mathtt{1f}$	$\mathtt{t2} \to \mathtt{2t}$	$\triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}$
$\mathtt{1t}\triangleright \to \mathtt{12}\triangleright$			
$\mathtt{2t}\triangleright \to \mathtt{22}\triangleright$			

Farkas' map (cont.)

$$[\mathtt{f}](x) = \begin{bmatrix} - & - & - & 2 & - \\ - & 2 & 0 & - & - \\ 2 & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} x \oplus \begin{bmatrix} 0 \\ - \\ - \\ - \\ - \end{bmatrix} \qquad [\mathtt{t}](x) = \begin{bmatrix} - & - & - & - & 2 \\ 0 & 2 & 0 & - & 0 \\ 2 & - & 2 & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} x \oplus \begin{bmatrix} 0 \\ - \\ - \\ - \\ - \end{bmatrix}$$

$$[\triangleleft](x) = \begin{bmatrix} 0 \\ 2 \\ - \\ - \\ 4 \end{bmatrix} \qquad [\triangleright](x) = \begin{bmatrix} 0 & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \\ - & - & - & - & - \end{bmatrix} x$$

$$[0](x) = \begin{bmatrix} 0 & 4 & 0 & - & - \\ - & 4 & - & - & - \\ - & 4 & 0 & - & - \\ 0 & 3 & 0 & - & - \\ - & - & - & - & - \end{bmatrix} x \qquad [1](x) = \begin{bmatrix} 1 & - & - & - & - \\ - & 4 & 0 & - & - \\ - & 4 & 0 & - & - \\ 0 & - & - & - & - \\ 0 & 3 & 0 & - & - \end{bmatrix} x \qquad [2](x) = \begin{bmatrix} 0 & - & 0 & - & - \\ - & 4 & - & - & - \\ 0 & - & 1 & - & 0 \\ - & - & - & - & - \\ 0 & - & 0 & - & 0 \end{bmatrix} x$$

Partial solution

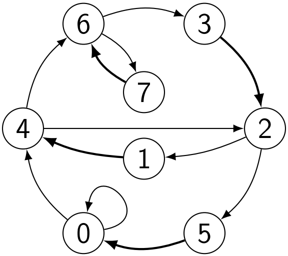
Leave out a rule from

$$
\begin{array}{llll}
\mathtt{f}\triangleright \to \triangleright & \mathtt{f0} \to \mathtt{0f} & \mathtt{t0} \to \mathtt{1t} & \triangleleft\mathtt{0} \to \triangleleft\mathtt{t} \\
\mathtt{t}\triangleright \to \mathtt{2}\triangleright & \mathtt{f1} \to \mathtt{0t} & \mathtt{t1} \to \mathtt{2f} & \triangleleft\mathtt{1} \to \triangleleft\mathtt{ff} \\
 & \mathtt{f2} \to \mathtt{1f} & \mathtt{t2} \to \mathtt{2t} & \triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}
\end{array}
$$

and try to prove termination.

All 11 subproblems are solvable.
$\Longrightarrow$ An interpretation that decreases strictly wrt any single rule (nonstrictly wrt others) settles the Collatz conjecture.

Collatz trajectories modulo 8



Nonconvergent trajectories cannot avoid $\{2, 3, 4, 6\}$.

It remains unknown whether any of $\{1, 5, 7\}$ has to be encountered in all trajectories.

SAT solving considerations

Phase-saving heuristic:

- Phase-saving uses cached assignments to variables when choosing a branch to explore.
- For small values, order encoding assigns a large fraction of the variables to false.
- Disable phase-saving and use negative branching: always explore the "false" branch first.

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Heavy-tailed behavior:

- There is a large variance in runtime across different initial conditions of the search. Some runs have a nonnegligible chance of finishing early.
- To take advantage, run multiple instances of the SAT solver with varying seeds and different shufflings of the formula.

Experiments with solver configurations

			Phase-saving		Negative branching	
Interpretation	*D*	*V*	Single	Parallel	Single	Parallel
Arctic	5	8	240.00s	240.00s	44.45s	**9.22s**
Arctic	3	5	1.52s	**0.13s**	29.95s	13.12s
Arctic	3	4	3.75s	**0.83s**	3.27s	1.71s
Natural	4	4	75.78s	19.12s	29.62s	**8.13s**
Natural	4	4	75.05s	**5.22s**	24.31s	6.43s
Arctic	4	3	3.33s	**0.52s**	11.55s	3.84s
Natural	3	11	240.00s	240.00s	240.00s	**79.05s**
Arctic	3	12	1.94s	**0.33s**	3.28s	0.38s

More problems

Mahler's $3/2$ **problem [1968].** Is there $\xi \in \mathbb{R}_{>0}$ such that $\operatorname{frac}\left(\xi\left(\frac{3}{2}\right)^k\right) < \frac{1}{2}$ for all $k \in \mathbb{N}$?

$$
\begin{array}{llll}
\texttt{f}\triangleright \to \texttt{0}\triangleright & \texttt{f0} \to \texttt{0f} & \texttt{t0} \to \texttt{1t} & \triangleleft\texttt{0} \to \triangleleft\texttt{t} \\
\texttt{ft}\triangleright \to \texttt{10}\triangleright & \texttt{f1} \to \texttt{0t} & \texttt{t1} \to \texttt{2f} & \triangleleft\texttt{1} \to \triangleleft\texttt{ff} \\
 & \texttt{f2} \to \texttt{1f} & \texttt{t2} \to \texttt{2t} & \triangleleft\texttt{2} \to \triangleleft\texttt{ft}
\end{array}
$$

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$\mathtt{f}\triangleright \to \mathtt{0}\triangleright$	$\mathtt{f0} \to \mathtt{0f}$	$\mathtt{t0} \to \mathtt{1t}$	$\triangleleft\mathtt{0} \to \triangleleft\mathtt{t}$
$\mathtt{ft}\triangleright \to \mathtt{10}\triangleright$	$\mathtt{f1} \to \mathtt{0t}$	$\mathtt{t1} \to \mathtt{2f}$	$\triangleleft\mathtt{1} \to \triangleleft\mathtt{ff}$
	$\mathtt{f2} \to \mathtt{1f}$	$\mathtt{t2} \to \mathtt{2t}$	$\triangleleft\mathtt{2} \to \triangleleft\mathtt{ft}$

Erdős on powers of 2 **[1979].** When does the ternary expansion of 2^n omit the digit 2?

$\mathtt{0}\triangleright \to \triangleright$	$\mathtt{0f} \to \mathtt{f0}$	$\mathtt{1t} \to \mathtt{t0}$	$\triangleleft\mathtt{t} \to \triangleleft\mathtt{0}$
$\mathtt{1}\triangleright \to \triangleright$	$\mathtt{0t} \to \mathtt{f1}$	$\mathtt{2f} \to \mathtt{t1}$	$\triangleleft\mathtt{ff} \to \triangleleft\mathtt{1}$
$\triangleleft\triangleright \to \triangleleft\triangleright$	$\mathtt{1f} \to \mathtt{f2}$	$\mathtt{2t} \to \mathtt{t2}$	$\triangleleft\mathtt{ft} \to \triangleleft\mathtt{2}$

Closing remarks

- Efficacy of techniques other than matrix interpretations
- Proving the *nonexistence* of matrix intepretations of any dimension for the Collatz system
- Moving from string to term rewriting
- Injecting specific knowledge about the problem into the rewriting system/termination proofs